

# Is there a relativistic nonlinear generalization of quantum mechanics?

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**Abstract.** Yes, there is. – A new kind of gauge theory is introduced, where the minimal coupling and corresponding covariant derivatives are defined in the space of functions pertaining to the functional Schrödinger picture of a given field theory. While, for simplicity, we study the example of a  $\mathcal{U}(1)$  symmetry, this kind of gauge theory can accommodate other symmetries as well. We consider the resulting relativistic nonlinear extension of quantum mechanics and show that it incorporates gravity in the (0+1)-dimensional limit, where it leads to the Schrödinger-Newton equations. Gravity is encoded here into a universal nonlinear extension of quantum theory. The probabilistic interpretation, i.e. Born’s rule, holds provided the underlying model has only dimensionless parameters.

## 1. Introduction

Linearity of the (functional) Schrödinger equation and the validity of the superposition principle have been essential ingredients of quantum (field) theory since its earliest days. Practically all physical phenomena behave nonlinearly when examined over a sufficiently large range of the dynamical parameters that determine their evolution. What singles out the linear dynamics for the wave function(al)? Quantum mechanics has been tested experimentally under a wide range of laboratory conditions and confirmed in all known cases. Yet the mathematical structure of the theory, so far, hinges heavily on its linearity embodied in linear operators acting on states represented by rays in a Hilbert space [1, 2].

This raises the question: Are nonlinear extensions possible which agree with the standard formulation in its experimentally ascertained domain of validity?

If so, could this alleviate the unresolved measurement problem [1, 2, 3]? While the outcome of this second question is still open, it seems worth while to mention that in recent studies of the related wave function collapse or reduction mechanisms by Pearle [4] and by Bassi [5] the authors indicate that a nonlinear extension of quantum theory, possibly involving additional degrees of freedom, might ultimately account consistently for these effects.

Our present aim is to report on a nonlinear extension of quantum field theory based on a new *functional gauge symmetry*, which operates on the space of field configurations rather than on the underlying spacetime [6]. In particular, we will argue that this theory essentially incorporates Newtonian gravity, which invites deliberation whether such an approach could be of wider use. Gravity, in this picture, appears as a manifestation of the nonlinearity of quantum mechanics.

Among the numerous earlier works that have attempted to extend quantum theory in a nonlinear way, there are: The work by Kibble and by Kibble and Randjbar-Daemi is close to

ours in that they consider how nonlinear modifications of quantum field theory can be made compatible with Lorentz or more generally coordinate invariance [7, 8]. Besides considering a coupling of quantum fields to classical gravity according to general relativity, which induces an intrinsic nonlinearity [8, 9], these authors study mean-field type nonlinearities, where parameters of the model are state dependent through their assumed dependence on expectations of certain operators. Work by Bialynicki-Birula and Mycielski introduces a logarithmic nonlinearity into the nonrelativistic Schrödinger equation, with which many of the features of standard quantum mechanics are left intact [10]. A number of different nonrelativistic models of this kind have been systematically studied by Weinberg, offering also an assessment of the observational limits on such modifications of the Schrödinger equation [11].

Independently, Doebner and Goldin and collaborators have also studied nonlinear modifications of the nonrelativistic Schrödinger equation [12]. This was originally motivated by attempts to incorporate dissipative effects. Later, however, they have shown that classes of nonlinear Schrödinger equations, including many of those considered earlier, can be obtained through nonlinear (in the wave function) transformations of the linear quantum mechanical equation. They coined the name “gauge transformations of the third kind” in this context, in analogy with gauge transformations of the second kind (corresponding to the usual minimal coupling). – In distinction, our functional gauge transformations work on the configuration space over which the *wave functional* is defined. This can be clearly seen in the way we introduce *covariant* functional derivatives (cf. Eqs. (12)–(13) in Section 3). (Of course, the functional derivatives here are to the functional Schrödinger picture of quantum field theory – reviewed (and generalized for fermions) in [13] – what ordinary derivatives are to quantum mechanics.)

The necessity of generalizing quantum dynamics for quantum gravity has been discussed in view of the “problem of time” and the Wheeler-DeWitt equation by Kiefer and by Barbour [14, 15]. – Note that this equation, playing the role of the Schrödinger equation there, is of the form of a constraint operator, i.e. the Hamiltonian of canonical gravity, acting on the wave functional,  $\hat{H}\Psi = 0$ . Two unpleasant features are incorporated here: no time derivative appears [9, 14] and, since  $\hat{H}$  is hermitean, nothing indicates the possibility of complex solutions [15]. – Both authors pointed out that nonlinear modifications could be a wellcome remedy and in Ref. [14] it was proposed that these might arise as a “supergauge potential” defined on configuration space. While formally analogous to the gauge connection in our covariant derivatives, however, only a preliminary interpretation in terms of certain quantum (vacuum) effects has been given.

Instead, based on the new functional gauge symmetry, all dynamical and constraint equations here will be derived from a gauge and Lorentz invariant action. *A priori* this has nothing to do with gravity, in particular, but may be applied to any quantum field theory.

The importance of a probabilistic interpretation of the wave function (“Born rule”) is emphasized in all previous works. We will recover this as well. However, no understanding of the origins of the proposed nonlinearities has been provided before, except in the case of semiclassical gravity studied by Kibble and Randjbar-Daemi [8]. This is achieved by our gauge principle, which, surprisingly, does incorporate a Newtonian form of gravity, see Section 6.

Part of the motivation for the present work comes from recent considerations of a possible deterministic foundation of quantum mechanics, as already verified in a number of models [16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. General principles and physical mechanisms ruling the construction of a deterministic classical model underlying a given quantum field theory are hard to come by, cf. Ref. [18]. However, the known toy models are promising, amounting to an existence proof – the quantum harmonic oscillator, for example, can be understood completely in classical deterministic terms, see Refs. [16, 17, 21].

We expect that with better understanding of the emergence of quantum mechanics also resulting nonlinear corrections to quantum mechanics should become visible. Models that are

based on linear (in the wave function) evolution equations alone presumably are not sufficient. Nonlinearity seems essential to go beyond the canonical framework of quantum theory. It is a central aspect in the following.

The paper is organized as follows. In Section 2, we recapitulate the work of Ref. [8], in order to argue that the gauge invariant (quantum) action introduced in Section 3 is Lorentz invariant, despite the presence of a fundamental length parameter. In Section 4, dynamical and constraint equations are presented and the crucial “nonlinearity factor” of the action is determined. In Section 5, we discuss the validity of the Born rule in the resulting nonlinear quantum theory. In the onedimensional limit, considering stationary states, it leads to the Schrödinger-Newton equations, see Section 6. Section 7 presents our concluding remarks.

## 2. Space-time and the Schrödinger picture

We briefly recall here the work of Kibble and Randjbar-Daemi [8]. Consider a four-dimensional globally hyperbolic manifold  $\mathcal{M}$  with a *given* metric  $g_{\mu\nu}$  of signature  $(1, -1, -1, -1)$ .<sup>1</sup> Then, it is possible to globally slice space-time into space-like hypersurfaces, such that a chosen family of such surfaces,  $\{\sigma(t)\}$ , is locally determined by:

$$x^\mu = x^\mu(\xi^1, \xi^2, \xi^3; t) , \quad (1)$$

in terms of intrinsic coordinates  $\xi^r$ , and there exists an everywhere time-like vectorfield  $n^\mu$ , the normal, with  $n_\mu n^\mu = 1$  and  $n_\mu x^\mu_{,r} = 0$ , where  $x^\mu_{,r} \equiv \partial x^\mu / \partial \xi^r$ . We will need the derivative with respect to  $t$  at fixed  $\xi^r$  of a function  $f$ ,  $\dot{f} \equiv \partial f / \partial t|_\xi$ . In particular, then, the lapse function  $N$  and shift vector  $N^r$  are introduced by  $\dot{x}^\mu = N n^\mu + N^r x^\mu_{,r}$ , the geometrical meaning of which is illustrated, for example, in Chapter 3.3 of reference [9].

We begin with a given Lagrangean  $L$  of a field theory, such as for a real scalar field  $\phi$ :

$$L \equiv \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) , \quad (2)$$

where  $V(\phi)$  incorporates mass or selfinteraction terms. The corresponding invariant action is:

$$S \equiv \int d^4x \sqrt{-g} L , \quad (3)$$

where  $g \equiv \det g_{\mu\nu}$ . This, in turn, yields the stress-energy tensor  $T^{\mu\nu}$ :

$$\frac{1}{2} \sqrt{-g} T^{\mu\nu} \equiv \frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{2} \sqrt{-g} (\partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} L) . \quad (4)$$

With the help of the induced metric  $\gamma_{rs}$  on  $\sigma(t)$ ,  $\gamma_{rs} \equiv g_{\mu\nu} x^\mu_{,r} x^\nu_{,s}$ , and the hypersurface element  $d\sigma_\mu \equiv d^3\xi \sqrt{-\gamma} n_\mu$ , the surface-dependent Hamiltonian can be defined:

$$H(t) \equiv \int_{\sigma(t)} d\sigma_\mu T^\mu_\nu \dot{x}^\nu . \quad (5)$$

In the simplest case, with  $\dot{x}^\mu = \delta^\mu_0$  (i.e.,  $N = 1$ ,  $N^r = 0$ ) and  $x^\mu_{,r} = \delta^\mu_r$ , these relations become  $\gamma_{rs} = g_{rs}$  and  $H(t) = \int_{\sigma(t)} d^3\xi T^{00}$ , as expected.

If the stress-energy tensor can be expressed in terms of canonical coordinates and momenta, say, the scalar field  $\phi = \phi(\xi^i, t)$  and its conjugated momentum  $\Pi = \Pi(\xi^i, t)$  on time slices  $\sigma(t)$ , we may assume that the corresponding quantized theory exists, with  $\phi$  and  $\Pi$  fulfilling

<sup>1</sup> We use units such that  $\hbar = c = 1$ .

standard equal- $t$  commutation relations. Matters are not that simple in a general curved background. Therefore, a heuristic derivation of the Schrödinger picture from the manifestly covariant Heisenberg picture has been presented in Ref. [8]. We will not pursue this, since our aim is only to recover their Lorentz invariant form of the functional Schrödinger equation, a generalization of which will follow from the action principle of the following section.<sup>2</sup>

In fact, the functional Schrödinger equation obtained by Kibble and Randjbar-Daemi appears naturally as one would guess:

$$i\dot{\Psi} = H(t)\Psi \quad . \quad (6)$$

Using the surface element  $d\sigma_\mu$  given above, together with Eq. (5), and:

$$\dot{\Psi} = \int_{\sigma(t)} d^3\xi \, \dot{x}^\mu \frac{\delta}{\delta x^\mu} \Psi \quad , \quad (7)$$

the Schrödinger equation can also be represented in a local form:

$$i \frac{\delta}{\delta x^\mu} \Psi = \sqrt{-\gamma} n_\nu T_\mu^\nu \Psi \quad . \quad (8)$$

Thus, the functional Schrödinger equation can be written in a way that makes explicit the behaviour under Lorentz transformations. We specialize to the case of a flat background space-time in the following, where field quantization is well understood.

### 3. A new action for a new gauge symmetry

We consider the generic scalar field theory described by the Lagrangean of Eq. (2), while internal symmetries and fermions can be introduced as we discussed earlier in the second of Refs. [6]. Furthermore, specializing the result of the previous section for *Minkowski space*, we find:

$$H(t) = \int_{\sigma(t)} d^3\xi \, T^{00} = \int d^3x \left\{ -\frac{1}{2} \frac{\delta^2}{\delta \phi^2} + \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right\} \equiv H[\hat{\pi}, \phi] \quad , \quad (9)$$

i.e., the usual Hamiltonian which is independent of the parameter time  $t$ ; intrinsic and Minkowski space coordinates have been identified,  $\vec{\xi} = \vec{x}$ . – Here, quantization is implemented by substituting the canonical momentum  $\pi$  conjugate to the field  $\phi$  (i.e. the “coordinate”):

$$\pi(\vec{x}) \longrightarrow \hat{\pi}(\vec{x}) \equiv \frac{1}{i} \frac{\delta}{\delta \phi(\vec{x})} \quad . \quad (10)$$

Correspondingly, we have  $\Psi = \Psi[\phi; t]$ , i.e. a time dependent functional, in this coordinate representation, and  $\dot{\Psi} = \partial_t \Psi$ . So far, this is the usual functional Schrödinger picture of quantum field theory applied to the example of a scalar model [13, 26].

Next, we introduce *functional gauge transformations* [6]:

$$\Psi'[\phi; t] = \exp(i\Lambda[\phi; t])\Psi[\phi; t] \quad , \quad (11)$$

where  $\Lambda$  denotes a time dependent real functional. These  $\mathcal{U}(1)$  transformations are local in the space of field configurations. They differ from the usual gauge transformations in QFT, since we introduce covariant derivatives by the following replacements:

$$\partial_t \longrightarrow \mathcal{D}_t \equiv \partial_t - i\mathcal{A}_t[\phi; t] \quad , \quad (12)$$

$$\frac{\delta}{\delta \phi(\vec{x})} \longrightarrow \mathcal{D}_{\phi(\vec{x})} \equiv \frac{\delta}{\delta \phi(\vec{x})} - i\mathcal{A}_\phi[\phi; t, \vec{x}] \quad . \quad (13)$$

<sup>2</sup> As remarked in Ref. [8], the derivation from an action principle guarantees the general coordinate invariance of the theory. However, the Schrödinger picture clearly depends on the slicing of space-time as well as on the parametrisation of the slices. Thus, invariance under surface deformations – which can be restricted to diffeomorphism invariance [9] – is not implied.

The real functional  $\mathcal{A}$  presents a new kind of ‘potential’ or ‘connection’. Generally,  $\mathcal{A}$  depends on  $t$ . However, it is a *functional* of  $\phi$  in Eq. (12), while it is a *functional field* in Eq. (13). We distinguish these components of  $\mathcal{A}$  by the subscripts. They are required to transform as:

$$\mathcal{A}'_t[\phi; t] = \mathcal{A}_t[\phi; t] + \partial_t \Lambda[\phi; t] , \quad (14)$$

$$\mathcal{A}'_\phi[\phi; t, \vec{x}] = \mathcal{A}_\phi[\phi; t, \vec{x}] + \frac{\delta}{\delta\phi(\vec{x})} \Lambda[\phi; t] . \quad (15)$$

Applying Eqs. (11)–(15), it follows that the correspondingly generalized functional Schrödinger equation is invariant under the  $\mathcal{U}(1)$  gauge transformations.

Furthermore, it is suggestive to introduce an invariant ‘field strength’:

$$\mathcal{F}_{t\phi}[\phi; t, \vec{x}] \equiv \partial_t \mathcal{A}_\phi[\phi; t, \vec{x}] - \frac{\delta}{\delta\phi(\vec{x})} \mathcal{A}_t[\phi; t] , \quad (16)$$

in close analogy to ordinary gauge theories; note that  $\mathcal{F}_{t\phi} = [\mathcal{D}_t, \mathcal{D}_\phi]/(-i)$ .

A consistent dynamics for the gauge ‘potential’  $\mathcal{A}$  has to be postulated, in order to give a meaning to the above ‘minimal coupling’ prescription. All elementary fields are present as the coordinates on which the wave functional depends – presently just a scalar field, besides time. We consider the following  $\mathcal{U}(1)$  invariant action:

$$\Gamma \equiv \int dt D\phi \left\{ \Psi^* \left( \mathcal{N}(\rho) \overleftrightarrow{i\mathcal{D}}_t - H \left[ \frac{1}{i} \mathcal{D}_\phi, \phi \right] \right) \Psi + \frac{l^2}{2} \int d^3x (\mathcal{F}_{t\phi})^2 \right\} , \quad (17)$$

where  $\Psi^* \mathcal{N} \overleftrightarrow{i\mathcal{D}}_t \Psi \equiv \frac{1}{2} \mathcal{N} \{ \Psi^* i\mathcal{D}_t \Psi + (i\mathcal{D}_t \Psi)^* \Psi \}$ , and with a dimensionless real function  $\mathcal{N}$  which depends on the density:

$$\rho[\phi; t] \equiv \Psi^*[\phi; t] \Psi[\phi; t] . \quad (18)$$

The function  $\mathcal{N}$  incorporates a *necessary nonlinearity*, which will be uniquely determined in Section 4, cf. Eq. (25). The fundamental parameter  $l$  has dimension  $[l] = [length]$ , for dimensionless measure  $D\phi$  and  $\Psi$ , independently of the dimension of space-time.

Our action  $\Gamma$  generalizes the one employed in Dirac’s variational principle for QFT, especially for a scalar field, in Refs. [8, 26]. The quadratic part in  $\mathcal{F}_{t\phi}$  is the simplest possible extension, i.e. local in  $\phi$  and quadratic in the derivatives, together with the nonlinearity  $\mathcal{N}(\rho)$ .

An immediate consequence of  $\mathcal{U}(1)$  invariance is that the Hamiltonian  $H$ , unlike in QFT, cannot be arbitrarily shifted by a constant  $\Delta E$ , gauge transforming  $\Psi \rightarrow \exp(-i\Delta E t) \Psi$ . Thus, there is an absolute meaning to the zero of energy in this theory.

Translation invariance of the action, Eq. (17), implies a *conserved energy functional*, where a contribution which is solely due to  $\mathcal{A}_t$  and  $\mathcal{A}_\phi$  is added to the matter term, which is modified by the covariant derivatives.

According to Section 2, the Lorentz invariance of this theory is guaranteed. The action can be written in a Lorentz (and Poincaré) invariant way, using the appropriate surface-dependent Hamiltonian, cf. Eq. (5), despite that a fundamental length  $l$  enters.<sup>3</sup>

The action depends on  $\Psi, \Psi^*, \mathcal{A}_t$ , and  $\mathcal{A}_\phi$ . The equations of motion and a constraint will be obtained by varying  $\Gamma$  with respect to these variables.

#### 4. The equations of motion and a constraint

The dynamical equations of motion are reproduced here for convenience, which were previously derived in Refs. [6]. The gauge covariant equation for the  $\Psi$ -functional is:

$$(\rho \mathcal{N}(\rho))' i\mathcal{D}_t \Psi[\phi; t] = H \left[ \frac{1}{i} \mathcal{D}_\phi, \phi \right] \Psi[\phi; t] , \quad (19)$$

<sup>3</sup> The coordinates  $x^\mu$ , of course, must not be confused with the intrinsic coordinates  $\xi^i$  and time parameter  $t$ .

where  $f'(\rho) \equiv df(\rho)/d\rho$ ; it replaces the usual functional Schrödinger equation.

The nonlinear Eq. (19) preserves the normalization of  $\Psi$ . Fixing it at an initial parameter time, in terms of an arbitrary constant  $\mathcal{C}_0$ :

$$\langle \Psi | \Psi \rangle \equiv \int D\phi \Psi^* \Psi = \mathcal{C}_0 , \quad (20)$$

it is conserved under further evolution, while the overlap of two different states,  $\langle \Psi_1 | \Psi_2 \rangle$ , may vary. This is a necessary ingredient of a probability interpretation related to  $\Psi^* \Psi$ , which will be discussed in more detail in the next section.

Completing the dynamical equations, there is an invariant ‘gauge field equation’:

$$\partial_t \mathcal{F}_{t\phi}[\phi; t, \vec{x}] = \frac{-1}{2il^2} \left( \Psi^*[\phi; t] \mathcal{D}_{\phi(\vec{x})} \Psi[\phi; t] - \Psi[\phi; t] (\mathcal{D}_{\phi(\vec{x})} \Psi[\phi; t])^* \right) . \quad (21)$$

However, there is no time derivative acting on the variable  $\mathcal{A}_t$  in the action. Therefore, it acts as a Lagrange multiplier for a constraint, which is the gauge invariant ‘Gauss’ law’:

$$\int d^3x \frac{\delta}{\delta \phi(\vec{x})} \mathcal{F}_{t\phi}[\phi; t, \vec{x}] = \frac{-1}{l^2} \rho \mathcal{N}(\rho) . \quad (22)$$

Of course, this differs from QED, for example, and raises the question, whether our functional  $\mathcal{U}(1)$  gauge symmetry is compatible with standard internal symmetries. This is answered affirmatively in the second of Refs. [6].

The Eq. (22) can be combined with Eq. (21) to result in a continuity equation:

$$0 = \partial_t \left( \rho \mathcal{N}(\rho) \right) - \frac{1}{2i} \int d^3x \frac{\delta}{\delta \phi(\vec{x})} \left( \Psi^* \mathcal{D}_{\phi(\vec{x})} \Psi - \Psi (\mathcal{D}_{\phi(\vec{x})} \Psi)^* \right) , \quad (23)$$

expressing local  $\mathcal{U}(1)$  ‘charge’ conservation in the space of field configurations. Functionally integrating Eq. (22), we find that the total ‘charge’  $Q$  has to vanish at all times:

$$Q(t) \equiv \frac{1}{l^2} \int D\phi \rho \mathcal{N}(\rho) = 0 , \quad (24)$$

since the functional integral of a total derivative is zero. The necessity of the nonlinearity now becomes obvious. Without it, the vanishing total ‘charge’ could not be implemented, as it would be in conflict with the normalization, Eq. (20).

Next, we determine the nonlinearity factor,  $\mathcal{N}(\rho) \neq 1$ . We would like to implement Eq. (24), similarly as the normalization, at an initial parameter time  $t$ . Since it has to be a constant of motion,  $\partial_t Q(t) = 0$ , we express this, with the help of Eq. (19), as a condition on  $\rho \mathcal{N}(\rho)$ . It is easily seen that the only solution here is a linear function:

$$\rho \mathcal{N}(\rho) = \mathcal{C}_1 \left( \rho - \mathcal{C}_0 \left( \int D\phi \right)^{-1} \right) , \quad (25)$$

if one wants to avoid further constraining  $\Psi$  or  $\Psi^*$ ; the latter would make it more difficult, if not impossible, to obtain linear quantum mechanics as a limiting case.<sup>4</sup>

Evidently, the volume of the space of fields,  $\Omega \equiv \int D\phi$ , needs to be regularized, as well as the second functional derivatives at coinciding points which appear. A cut-off on field amplitudes has to be introduced together, for example, with the point-splitting technique [13]. A related renormalization procedure is an interesting subject for further study, taking into account the new functional gauge symmetry.

<sup>4</sup> In Ref. [6], we used a logarithmic function; it has to be discarded, since it is not related to a constant of motion.

## 5. Interpreting $\Psi^*\Psi$

The probability interpretation of the density  $\rho = \Psi^*\Psi$  (Born's rule) can be applied, if the *homogeneity property* holds [7, 10, 11]:  $\Psi$  and  $z\Psi$  ( $z \in \mathbf{Z}$ ) represent the same physical state. Thus, states are associated with rays in a Hilbert space (instead of vectors).

In order to investigate the present case, it is useful to consider scale transformations:

$$\rho = \mathcal{C}_0^a \mathcal{C}_1^{-1} \rho' \quad , \quad \int D\phi = \mathcal{C}_0^{1-a} \mathcal{C}_1 \int D\phi' \quad , \quad (26)$$

such that  $\int D\phi' \rho' = 1$ ; we recall that the real measure  $D\phi$  and constants  $\mathcal{C}_{0,1}$  are chosen dimensionless, without loss of generality;  $a$  is real. Furthermore, we rescale:

$$(\vec{x}; t) = \mathcal{C}_0^{-a/2} \mathcal{C}_1^{1/2} (\vec{x}'; t') \quad , \quad (\phi; \mathcal{A}_t) = \mathcal{C}_0^{a/2} \mathcal{C}_1^{-1/2} (\phi'; \mathcal{A}'_t) \quad , \quad (27)$$

and, consistently:

$$(\delta_\phi; \mathcal{A}_\phi) = \mathcal{C}_0^a \mathcal{C}_1^{-1} (\delta_{\phi'}; \mathcal{A}_{\phi'}) \quad . \quad (28)$$

Under these transformations, the action transforms as:

$$\Gamma = \mathcal{C}_1 \Gamma' \quad , \quad (29)$$

where  $\Gamma'$  is defined like  $\Gamma$ , Eq. (17), however, replacing all quantities by the primed ones. To arrive at this result, the Hamiltonian  $H$ , cf. Eq. (9), must *not contain dimensionfull constants*.

There are several implications. – First, the scale transformations change the overall scale of the action, say, in units of  $\hbar$ , by the constant factor  $\mathcal{C}_1$ . This is equivalent to the rescaling  $\hbar = \hbar'/\mathcal{C}_1$ . However, since we prefer to choose units such that  $\hbar = 1$ , we should also fix  $\mathcal{C}_1 = 1$ , henceforth. constant  $\mathcal{C}_0$  does not affect the transformation of  $\Gamma$ , we can always choose to normalize the wave functional to  $\mathcal{C}_0 = 1$ , see Eq. (20).

In this way, we see that states, as far as  $\Psi$  is concerned, are represented by rays. Therefore, a probability interpretation of  $\Psi^*\Psi$  according to the *Born rule* can be maintained. This is in agreement with the observation that Eq. (19), if it were not for the presence of the covariant derivatives, now appears like the usual functional Schrödinger equation. Summarizing the previous discussion, we have:

$$\rho \mathcal{N}(\rho) = \rho - \left( \int D\phi \right)^{-1} \quad , \quad (30)$$

$$i\mathcal{D}_t \Psi[\phi; t] = H\left[\frac{1}{i}\mathcal{D}_\phi, \phi\right] \Psi[\phi; t] \quad . \quad (31)$$

However, it must be stressed that the ‘potentials’  $\mathcal{A}_t$  and  $\mathcal{A}_\phi$  are selfconsistently determined through Eqs. (21)–(22). Therefore, we arrive here at intrinsically *nonlinear quantum mechanics*.<sup>5</sup>

The difference to standard quantum mechanics also shows up clearly in Eq. (23), with the first term now replaced by  $\partial_t \rho$ : the flux of probability over the space of field configurations is affected nonlinearly by  $\Psi^*$  and  $\Psi$  through the ‘potential’  $\mathcal{A}_\phi$ .

Finally, we remark that *in presence of dimensionfull parameters* in the Hamiltonian the above scale symmetry, Eqs. (26)–(29), breaks down. Then, the normalization of  $\Psi$  cannot be chosen

<sup>5</sup> In the second of Refs.[6], we have argued that *microcausality* of the present theory holds. – The *weak superposition principle* [10], generally, must be expected to fail: for two non-overlapping sources adding to the right-hand sides of Eqs. (21)–(22), the resulting ‘potentials’ must be expected to propagate away from the sources in field space. Thus, the sum of two non-overlapping solutions  $\Psi_{1,2}$  will hardly present a solution of the coupled equations. However, if two *stationary non-overlapping solutions* exist, then their sum also presents a solution; see the stationary equations in Section 6.

freely, i.e., rays break into inequivalent vectors. In this situation, it is appropriate to consider  $\Psi$  and  $\Psi^*$  as giving rise to two oppositely ‘charged’ real components of the wave functional,  $\Psi_+ \equiv (\Psi + \Psi^*)/\sqrt{2}$  and  $\Psi_- \equiv (\Psi - \Psi^*)/i\sqrt{2}$ , which interact, while preserving the normalization of  $\Psi^*\Psi$ . Different normalizations, then, correspond to physically different sectors of the theory.

The *absence of the homogeneity property* modifies the usual measurement theory. In particular, the usual “reduction of the wave packet” postulate [1] cannot be maintained. This case has been discussed in detail in Ref. [7] and formed the starting point for the particular nonlinear theory proposed there, mentioned before in Section 1.

## 6. Stationary states and Schrödinger-Newton equations

The time dependence in Eqs. (19)–(22) can be separated with the Ansatz  $\Psi[\phi; t] \equiv \exp(-i\omega t)\Psi_\omega[\phi]$ ,  $\omega \in \mathbf{R}$ , and consistently assuming that the  $\mathcal{A}$ -functionals are *time independent*. Thus, the Eq. (19), together with Eq. (30), yields:

$$\omega\Psi_\omega[\phi] = H\left[\frac{1}{i}\mathcal{D}_\phi, \phi\right]\Psi_\omega[\phi] - \mathcal{A}_t[\phi]\Psi_\omega[\phi] \quad , \quad (32)$$

with  $\mathcal{D}_\phi = \frac{\delta}{\delta\phi} + i\mathcal{A}_\phi$  and  $\rho_\omega \equiv \Psi_\omega^*[\phi]\Psi_\omega[\phi]$ . We obtain from Eq. (21):

$$\frac{1}{2i} \left( \Psi_\omega^*[\phi]\mathcal{D}_{\phi(\vec{x})}\Psi_\omega[\phi] - \Psi_\omega[\phi](\mathcal{D}_{\phi(\vec{x})}\Psi_\omega[\phi])^* \right) = 0 \quad , \quad (33)$$

which expresses the vanishing of the ‘current’ in the stationary situation. – Applying a time independent gauge transformation, cf. Eqs. (11), (15), the stationary wave functional can be made *real*. Then, the Eq. (33) implies  $\mathcal{A}_\phi = 0$ ; consequently,  $\mathcal{D}_\phi \rightarrow \frac{\delta}{\delta\phi}$  everywhere. Finally, ‘Gauss’ law’, Eq. (22), determines  $\mathcal{A}_t$ :

$$\int d^3x \frac{\delta^2}{\delta\phi(\vec{x})^2} \mathcal{A}_t[\phi] = \frac{1}{l^2} \left( \rho - (\int D\phi)^{-1} \right) \quad , \quad (34)$$

which has to be solved selfconsistently together with Eq. (32). – Thus, separation of the time dependence has given us two coupled equations. They represent a field theoretic generalization of the stationary Schrödinger-Newton equations, as we shall now explain.

The time dependent *Schrödinger-Newton equations* for a particle of mass  $m$  are given by:

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi - m\Phi\psi \quad , \quad \nabla^2\Phi = 4\pi Gm|\psi|^2 \quad , \quad (35)$$

where  $G \equiv l_P^2 c^2 / \hbar$  is Newton’s gravitational constant (here related to the Planck length  $l_P$ ) and  $\Phi$  denotes the gravitational potential. They represent the nonrelativistic approximation to “semiclassical gravity”, i.e. Einstein’s field equations coupled to the expectation value of the operator-valued stress-energy tensor of quantum matter. They are considered in arguments related to “semiclassical gravity”, to gravitational self-localization of mesoscopic or macroscopic mass distributions, and to the role of gravity in the objective reduction scenarios of Diósi and of Penrose – see, for example, the Refs. [8, 27, 28, 29, 30, 31], and further references therein.

In a Universe which consists only of a single point, our field theory equations (32) and (34) indeed reduce to the stationary Schrödinger-Newton equations *in one dimension*. Appropriate rescalings by powers of  $l$ ,  $m$ ,  $\hbar$ , and  $c$  of the various quantities have to be incorporated, in order to give the equations their onedimensional form. With a nonzero potential  $V(\phi)$  in our Hamiltonian, the Schrödinger equation in (35) would acquire an additional term.

More generally, the Schrödinger-Newton equations present the formal limit of the present gauge theory in 0+1 dimensions, i.e. the quantum mechanical limit related to the usual



discussions of these equations. This can be generalized by considering a (lattice) discretized version of the functional equations, which then amounts to a quantum many-body theory incorporating a form of gravity.

It seems remarkable that the gravitational interaction arises here in the space of quantum states (configuration space). Yet, in view of the fundamental length  $l$  present in the action, Eq. (17), it is not a complete surprise that our gauge theory incorporates gravity. We notice, however, also a *deviation from Newtonian gravity*, presented by the constant term on the right-hand side of Eq. (34). While it is natural to let this term become arbitrarily small in the quantum mechanical limit just discussed, its presence is necessary for the full theory, cf. Section 4. Thus, gravity can turn from being attractive to being repulsive, depending on whether the right-hand side of this equation is negative or positive, respectively.

In Ref. [30], it has recently been shown that sufficiently large Gaussian wave packets show a tendency to shrink in width as they evolve according to the time dependent Schrödinger-Newton equations. This leads to a *decrease of interference effects*, which possibly will be observable in near-future molecular interference experiments. We speculate that according to the present theory coherent superpositions of displaced wave packets (*Schrödinger cat states*) decay by giving rise to time dependent ‘potentials’  $\mathcal{A}_t$  and  $\mathcal{A}_\phi$ , while attracting each other similar to corresponding classical matter distributions. Such behaviour could have some impact on dynamical ‘collapse of the wave function’ or reduction theories.

## 7. Conclusions

A relativistic  $\mathcal{U}(1)$  gauge theory has been presented which constitutes an intrinsically nonlinear extension of quantum mechanics or quantum field theory.

Closest in spirit is the work of Kibble and Randjbar-Daemi [8] where such nonlinearities – due to coupling the expectation of the matter stress-energy tensor to classical general relativity or due to making parameters of the model state dependent – have been discussed in a relativistic setting before. However, this has been reminiscent of a mean-field approximation.

In distinction, based on a new gauge principle, we have introduced two ‘potentials’,  $\mathcal{A}_t$  and  $\mathcal{A}_\phi$ , which are *not* additional independent fields but functionals that depend on the same field variables of the underlying (scalar or other) field theory as the wave functional  $\Psi$ . Their dynamical and constraint equations follow from a relativistic invariant action principle, introduced in Section 3. Thus, if the ‘potentials’ are eliminated by solving the respective equations, in principle, a nonlinear theory in  $\Psi$  necessarily results.

Note that in the absence of quantum matter,  $\Psi = 0$ , the Eqs. (21) and (22) that determine the ‘field strength’  $\mathcal{F}_{t\phi}$  – and similarly in the (0+1)-dimensional limit – have no time dependent solutions. Therefore, the ‘potentials’ do not propagate independently of matter sources here.<sup>6</sup>

We have shown that the homogeneity property holds which is necessary for the representation of states by rays in Hilbert space. Thus, the Born rule can be applied, giving a probabilistic interpretation to  $\Psi^*\Psi$  [7, 10, 11]. However, it breaks down, if the assumed underlying classical model contains dimensionfull parameters. In this case, a discussion in terms of the ‘charged’ components of  $\Psi$  is appropriate, which invites further interpretation.

Related to the presence of a fundamental length  $l$  in the action, in the zerodimensional limit the presented theory recovers the Schrödinger-Newton equations, coupling Newtonian gravity to quantum mechanics [8, 27, 28, 29, 30, 31]. Thus, the proposed theory incorporates Newtonian gravity into quantum field theory: *unlike the standard coupling of independent gravitational degrees of freedom to matter, gravity is encoded here into a universal nonlinear extension of quantum field theory.*

<sup>6</sup> This is due to the fact that the analogue of a magnetic field is missing for any underlying model based on a one-component field, see Eqs. (2) and (16). The situation changes in the presence of internal symmetries, as discussed in the second of Refs. [6].

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